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Dynamic Response Analysis of Large Latticed Space Structures

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Abstract: In this paper an alternative method for dynamic response analysis of large space structures is presented, for which conventional finite element analysis would require excessive computer storage and computational time. Latticed structures in which the height is very small in comparison to its overall length and width are considered. The method is based on the assumption that the structure can be embedded in its continuum, in which any fiber can translate and rotate without deforming. An appropriate kinematically admissible series function is constructed to describe the deformation of the middle plane of this continuum. The unknown coefficients in this function are called the degree-of-freedom of the continuum, which is given the name "super element." Transformation matrices are developed to express the equations of motion of the actual systems in terms of the degrees-of-freedom of the super element. Thus, by changing the number of terms in the assumed function, the degrees-of-freedom of the super element can be increased or decreased. The super element response results are transformed back to obtain the desired response results of the actual system. The method is demonstrated for a structure woven in the shape of an Archimedian spiral.

1. Introduction

The introduction of the Space Station concept for having a permanent residence in space is a notable step in man's push towards a more effective use of space to advance his knowledge, improve his well-being, and to satisfy his drive to explore the unknown. The functions of a Space Station would be to serve as a scientific laboratory, a construction base from which to build and assemble large systems, and a site from which it will be possible to launch those large systems into higher orbits or beyond.¹ The Manned Maneuvering Unit has been used for the deployment, assembly and repair of structures in space during various missions of the space shuttle, and it is envisioned that the space shuttle would be used for the initial assembly of the

Space Station and other structures such as solar power stations, large space mirrors, telescopes and multipurpose large space platforms. Logsdon and Butler² have presented a historical review of various proposed space station concepts. The dynamic response analysis of such structures by conventional finite element methods involves stiffness and mass matrices of very large sizes which would require excessive computer storage and computational time. Also, since the response analysis procedures adopted for such problems require solutions of very large number of linear equations, round-off errors creep in at the end of each iteration, and these may be very significant at the end of a large number of iterations.

As an alternative, a method is presented in this paper in which the whole structural system is

assumed to be embedded in its continuum. The method consists of constructing an appropriate kinematically admissible trigonometric function for the middle plane of this continuum. The unknown coefficients in this function are the generalized coordinates or the degrees-of-freedom of the continuum, which is called the "super element". By changing the number of terms in this assumed trigonometric series, the degrees-of-freedom of the super element can be increased or decreased. Appropriate transformation matrices are developed to transform the displacements of each node of the actual system to conform to the degrees-of-freedom of the super element. This transformation matrix is used to relate the degrees-of-freedom of a typical beam element of the structure to the degrees-of-freedom of the super element. Thus, for each beam element of the structure, the stiffness and the mass matrices are assembled in terms of the super element degrees-of-freedom. The total stiffness and mass matrices

are obtained by algebraically adding the contribution of each element. The resulting equation of motion for the super element can be solved by using the modal superposition method or any step-by-step integration method. The method presented is demonstrated for analysis of a large latticed structure in the shape of an Archimedian spiral.

2. Formulation

Analytical Model and Basic Concepts

The particular structure considered is an assemblage of three-dimensional beam elements connected at discrete points (called nodes) such that the whole system is symmetrical about a plane (called middle plane) passing through the mid-height. The overall system geometry is taken such that the height dimension is very small in comparison to its overall length and width dimensions (which describe the geometry of the middle plane).

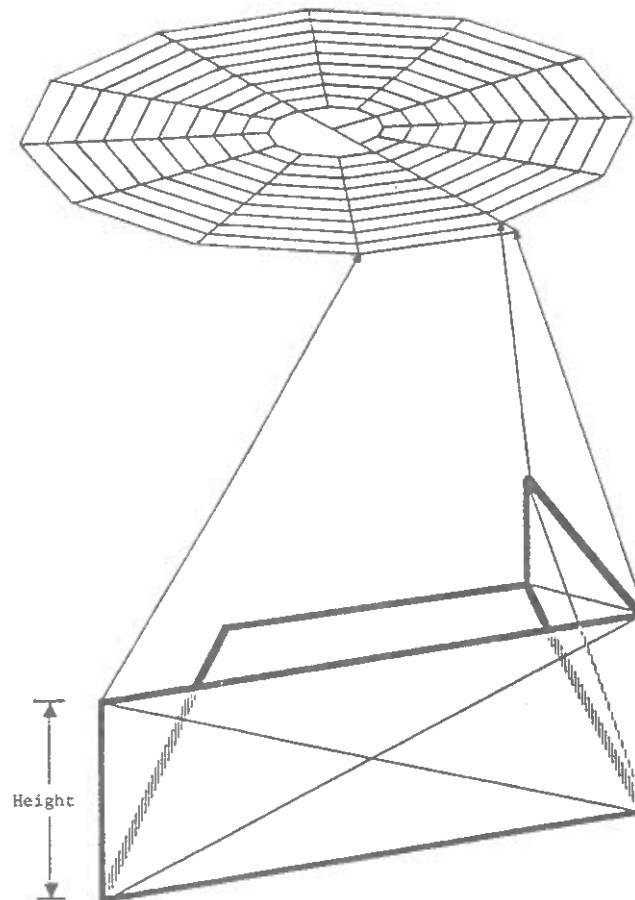


Fig. 1. A typical layout of the solar panel structure.

Figure 1 represents such a large lattice structure woven in the shape of an Archmedian spiral and fabricated from very thin aluminum sheets. Cutouts are made in this sheet to fabricate the framed structure, and which can be used to install solar cells in the empty spaces so created. Considering such a structural system with nineteen loops and a minimum and maximum radius of 5 ft and 325 ft, respectively, will result in a discretized finite element model with around 150,000 degrees-of-freedom. Dynamic response analysis of such a model would require excessive computer storage and computational time.

To develop an alternative dynamic response procedure for analysis of such a structure, it is assumed that the whole structural system is embedded in its continuum. In this continuum, any fiber parallel to the height dimension is assumed to translate and rotate without deforming. This actually implies that in the imaginary continuum, plane sections perpendicular to the middle plane before deformation remain plane and perpendicular to the deformed middle plane. This assumption is the same as the one used in developing the bending theory of thin plates. Thus, the equation of the deformed middle plane can be used to uniquely determine the state of stress at any point in the imaginary continuum at any instant. An appropriate kinematically admissible function is constructed to describe the deformation of the middle plane. This imaginary continuum is referred to as the "pseudo system." Any point in this pseudo system has six kinematic degrees-of-freedom. These six degrees-of-freedom are actually the deformations of the corresponding projected point on the middle plane and consist of three translations (along the global x-, y- and z-axis) and the three rates of change of these translations with respect to the global axis which is parallel to the height dimension (here this axis is assumed to be the z-axis).

It is assumed that the material remains in the linear elastic range and that all deformations are small (i.e., small deformation theory is valid). The deformations are divided into two categories: in-plane and out-of-plane. The in-plane deformations include the deformations in the plane of the middle plane, whereas, the out-of-plane deformations are deformations perpendicular to the

middle plane. The middle plane deformations are expressed in terms of a trigonometric series in cylindrical coordinates. The unknown coefficients in this trigonometric series are treated as the unknown degrees-of-freedom (i.e., generalized coordinates) of the pseudo system. Thus, the imaginary continuum or the pseudo system can be visualized as one "super element". The analysis procedure consists of first developing a linear transformation matrix relating the (12×12) stiffness and mass matrices of a typical beam element of the structure to the degrees-of-freedom of the super element. Then, this transformation matrix is used to find the contribution of the stiffness and mass matrices of each beam element of the actual system to the corresponding matrices of the super element. In this manner the stiffness and mass matrices of the super element are obtained by algebraically summing the contribution due to each beam element. All beam elements connected to a node with fixed boundary constraints are modelled such that the fixed element degrees-of-freedom are condensed out in the element stiffness and mass matrices. Thus, in its present form this method can only be applied when the boundary displacements are equal to zero (i.e., fixed boundary conditions).

It is assumed that the structure is subjected to only prescribed dynamic loads. For transient response analysis, first the response of the super element due to external dynamic loads is determined. Any response analysis technique, depending on the given load history, can be used to obtain the displacement, the velocity and the acceleration variations of this system with respect to time. Then, these are transformed back to the actual system using the linear transformation matrices developed for each element. It should be noted that, since the in-plane and the out-of-plane degrees-of-freedom are treated separately, the in-plane and the out-of-plane transient response analyses can be conducted separately.

In the analytical method presented, the size of the problem analyzed is governed by the number of degrees-of-freedom considered for the super element, which are actually the number of terms considered in the assumed trigonometric series used for the description of the middle plane deformations of the pseudo system. The number of

terms needed to accurately model this behavior is envisaged to be much less than the total degrees-of-freedom of the actual discretized finite element model. Also, since the in-plane and the out-of-plane motions are analyzed separately, this results in additional savings in computational time.

In summary, the method requires the following relationships to be developed:

- (1) Transformation relating the degrees-of-freedom of a beam element to the degrees-of-freedom of the super element.
- (2) Transformed mass and stiffness matrices of a beam element in terms of the degrees-of-freedom of the super element.

- (3) Transformed external dynamic force vector of the actual structural system in terms of the degrees-of-freedom of the super element.

The formulation of these relationships is presented first, which is followed by the response solution procedure and a numerical example.

Transformation Matrix for a Beam Element

A typical three-dimensional beam element is shown in Fig. 2. The \bar{x} -axis is taken as the centroidal axis of the beam member and the \bar{y} -axis and the \bar{z} -axis are taken as the principal axes of the beam

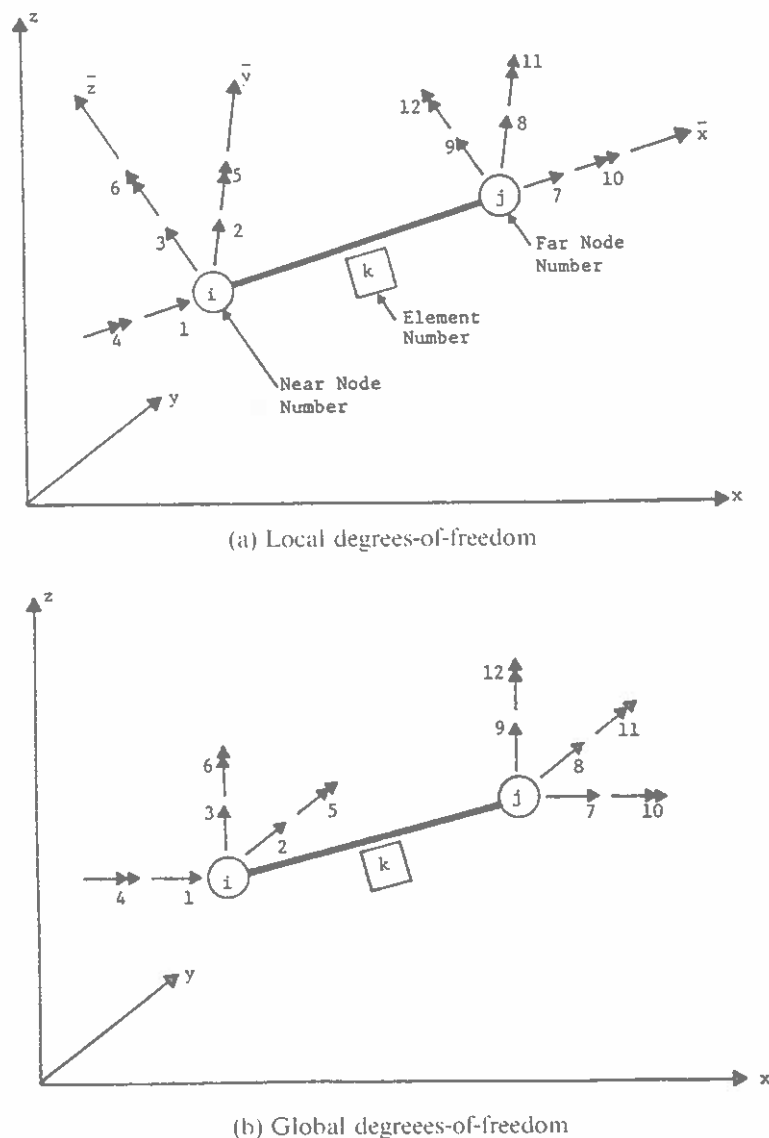


Fig. 2. Degrees-of-freedom numbers of a three-dimensional beam element.

cross-section. The degrees-of-freedom numbers of the beam element with respect to the local element axes and the global axes, respectively, are shown in Figs. 2(a) and 2(b). In these figures, translation is represented by an arrow head vector, whereas rotation about an axis is represented by a double arrow head vector in the direction of the axis.

The analytical development for the transformation matrix relating the degrees-of-freedom of the beam element to the fictitious degrees-of-freedom assumed for the super element involves the following steps:

(1) Development of the relationship (called

“nodal transformation”) between the displacement components of a node of the actual system to the assumed displacement function for the middle plane of the pseudo continuum or the super element.

(2) Applying the aforementioned relationship to the near end and the far end nodal degrees-of-freedom of the beam element and forming the desired “beam element transformation.”

The development of these transformation matrices is presented in following sub-sections.

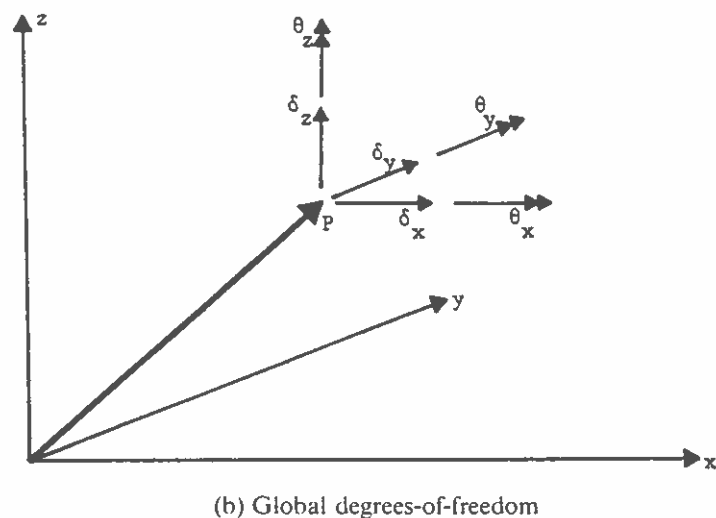
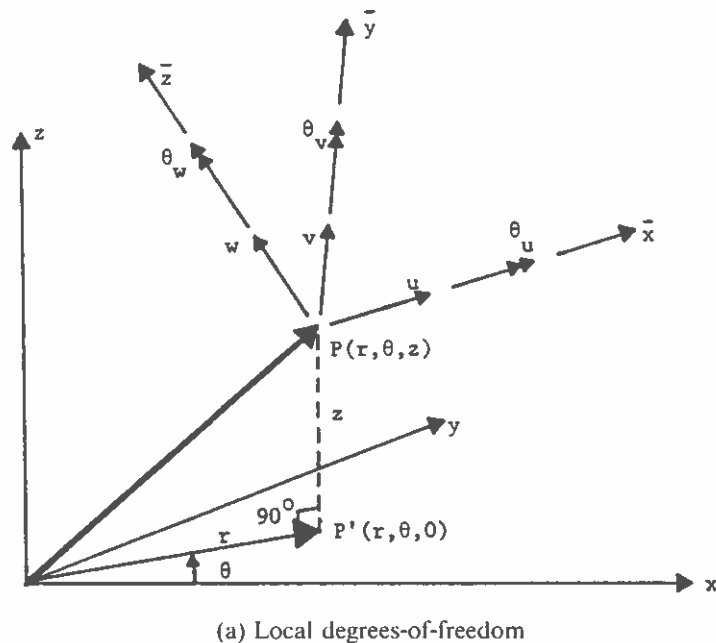


Fig. 3. Local and global degrees-of-freedom at a node

(1) Nodal Transformation

Consider a node **P** of the actual system (i.e., node of a beam element), represented in cylindrical coordinates (r, θ, z) , as shown in Fig. 3. This node undergoes translations u, v and w , and rotations θ_u, θ_v and θ_w , respectively, parallel to the local axes, as shown in Fig. 3(a). These deformations of node **P** can be transformed to translations δ_x, δ_y and δ_z , and rotations θ_x, θ_y and θ_z , respectively, parallel to the global axes, as shown in Fig. 3(b) and expressed by

$$\{\delta\} = [T] \{u\} \quad (1a)$$

$$\{\theta\} = [T] \{\bar{\theta}\} \quad (1b)$$

where

$$\{\delta\} = \begin{Bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{Bmatrix}; \{u\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}; \{\theta\} = \begin{Bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix};$$

$$\{\bar{\theta}\} = \begin{Bmatrix} \theta_u \\ \theta_v \\ \theta_w \end{Bmatrix}; \text{ and } [T] = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)$$

In Eq. (2), matrix $[T]$ is represented in a succinct fashion by using c and s to abbreviate $\cos\theta$ and $\sin\theta$, respectively.

Since fibers oriented parallel to the z -axis translate and rotate but do not deform, thus in cylindrical coordinates, the local displacement at any point, **P**, with coordinates (r, θ, z) , can be related to the displacements of the middle plane of the pseudo continuum by the following expressions:

$$u(r, \theta, z) = u(r, \theta, 0) + z\alpha(r, \theta, 0) \quad (3a)$$

$$v(r, \theta, z) = v(r, \theta, 0) + z\beta(r, \theta, 0) \quad (3b)$$

$$w(r, \theta, z) = w(r, \theta, 0) + z\tau(r, \theta, 0) \quad (3c)$$

where α, β and τ are the gradients given by

$$\alpha = \frac{\partial u}{\partial z}; \quad \beta = \frac{\partial v}{\partial z}; \quad \text{and } \tau = \frac{\partial w}{\partial z} \quad (4)$$

and $u(r, \theta, 0), v(r, \theta, 0)$ and $w(r, \theta, 0)$ are the displacement fields of the middle surface of the pseudo continuum or the super element.

Substituting expressions for u, v and w obtained in Eqs. (3a), (3b) and (3c) into Eq. (1a), gives

$$\{\delta\} = [T] [A] \{u\} \quad (5)$$

where

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & z & 0 \\ 0 & 1 & 0 & 0 & 0 & z \\ 0 & 0 & z & 1 & 0 & 0 \end{bmatrix} \quad (6a)$$

$$\{u\} = \begin{Bmatrix} u(r, \theta, 0) \\ v(r, \theta, 0) \\ \tau(r, \theta, 0) \\ \text{-----} \\ w(r, \theta, 0) \\ \alpha(r, \theta, 0) \\ \beta(r, \theta, 0) \end{Bmatrix} \quad (6b)$$

in which u, v, w, α, β and τ are functions of r and θ . It should be noted that in Eq. (6b) the in-plane and the out-of-plane displacement components with respect to the middle plane are partitioned separately. These displacement components will be represented by a truncated trigonometric series of the following form:

$$f(r, \theta) = \frac{fr}{r_{\max}} \sum_{i=1}^n (f_{ai} + \sin i\theta + f_{bi} \cos i\theta) \quad (7)$$

where f is the ratio of the degrees-of-freedom of the actual structure to the degrees-of-freedom given to the super element ($=n$); r_{\max} denotes the maximum value of r in the structure; n is an integer which gives the number of terms up to which truncation is carried out; and f_{a1} through f_{an} and f_{b1} through f_{bn} represent the unknown coefficients of the trigonometric series. These coefficients define the generalized coordinates, which are treated as the

fictitious degrees-of-freedom of the super element. In matrix notation, Eq.(7) can be rewritten as

$$f(r, \theta) = \{\eta\}^T \{\epsilon\} \quad (8)$$

where superscript T denotes the transpose of a matrix, and

$$\{\epsilon\}^T = \langle f_{a1} f_{a2} \dots f_{an} \mid f_{b1} f_{b2} \dots f_{bn} \rangle$$

$$= \langle \langle F_a \rangle \mid \langle F_b \rangle \rangle ; \text{ and} \quad (9a)$$

$$\{\eta\}^T = \frac{fr}{r_{\max}} \langle \sin \theta \sin 2\theta \dots \sin n\theta \mid \cos \theta \cos 2\theta \dots \cos n\theta \rangle \quad (9b)$$

In Eqs. (9a) and (9b), a row matrix is denoted by $\langle \rangle$.

Using the notation followed in Eqs. (8), (9a) and (9b) for u, v, τ , w, α and β , the middle plane displacement components are represented as

$$u\{r, \theta, 0\} = \{\eta\}^T \{\epsilon_u\} \quad (10a)$$

$$v\{r, \theta, 0\} = \{\eta\}^T \{\epsilon_v\} \quad (10b)$$

$$\tau\{r, \theta, 0\} = \{\eta\}^T \{\epsilon_\tau\} \quad (10c)$$

$$w\{r, \theta, 0\} = \{\eta\}^T \{\epsilon_w\} \quad (10d)$$

$$\alpha\{r, \theta, 0\} = \{\eta\}^T \{\epsilon_\alpha\} \quad (10e)$$

$$\beta\{r, \theta, 0\} = \{\eta\}^T \{\epsilon_\beta\} \quad (10f)$$

where $\{\epsilon_u\}$ denotes the unknown vector $\{\epsilon\}$ (refer to Eq. (9a)) pertaining to the u-displacement field, etc. Substituting these into Eq. (6b), gives

$$\{u\} = [N]^T \{E\} \quad (11)$$

where

$$[N] = \begin{bmatrix} \{\eta\}^T & 0 & 0 & 0 & 0 & 0 \\ 0 & \{\eta\}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & \{\eta\}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & \{\eta\}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & \{\eta\}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & \{\eta\}^T \end{bmatrix} \quad (12a)$$

$$\{E\}^T = \langle \{\epsilon_u\}, \{\epsilon_v\}, \{\epsilon_\tau\} \mid \{\epsilon_w\}, \{\epsilon_\alpha\}, \{\epsilon_\beta\} \rangle \quad (12b)$$

Substituting Eq. (11) into Eq. (5), yields

$$\{\delta\} = [[R_{11}] \mid [R_{12}]] \{E\} \quad (13)$$

where

$$[R_{11}] = \begin{bmatrix} c\{\eta\}^T & -s\{\eta\}^T & 0 \\ s\{\eta\}^T & c\{\eta\}^T & 0 \\ 0 & 0 & z\{\eta\}^T \end{bmatrix} \quad (14a)$$

$$[R_{12}] = \begin{bmatrix} 0 & cz\{\eta\}^T & -sz\{\eta\}^T \\ 0 & sz\{\eta\}^T & cz\{\eta\}^T \\ \{\eta\}^T & 0 & 0 \end{bmatrix} \quad (14b)$$

The rotations are related to the components of the middle plane displacements of the pseudo system by the following expressions:

$$\theta_u = \frac{1}{2r} (w_\theta + z\tau_\theta) - \frac{1}{2} \beta \quad (15a)$$

$$\theta_v = \frac{1}{2r} \alpha - \frac{1}{2} (w_r + z\tau_r) \quad (15b)$$

$$\theta_w = \frac{1}{2r} (v + z\beta) + \frac{1}{2} (v_r + z\beta_r) - \frac{1}{2r} (u_\theta + z\alpha_\theta) \quad (15c)$$

where

$$u_\theta = \frac{\partial u}{\partial \theta} ; v_r = \frac{\partial v}{\partial r} ; w_r = \frac{\partial w}{\partial r} ; w_\theta = \frac{\partial w}{\partial \theta} ;$$

$$\alpha_\theta = \frac{\partial \alpha}{\partial \theta} ; \beta_r = \frac{\partial \beta}{\partial r} ; \tau_r = \frac{\partial \tau}{\partial r} ; \tau_\theta = \frac{\partial \tau}{\partial \theta} ; \quad (16)$$

Substituting Eqs. (10a) through (10f) into Eq. (16), gives

$$u_\theta = \{\eta_\theta\}^T \{\epsilon_u\} \quad (17a)$$

$$v_r = \{\eta_r\}^T \{\epsilon_v\} \quad (17b)$$

$$w_r = \{\eta_r\}^T \{\epsilon_w\} \quad (17c)$$

$$w_\theta = \{\eta_\theta\}^T \{\epsilon_w\} \quad (17d)$$

$$\alpha_\theta = \{\eta_\theta\}^T \{\epsilon_\alpha\} \quad (17e)$$

$$\beta_r = \{\eta_r\}^T \{\epsilon_\beta\} \quad (17f)$$

$$\tau_r = \{\eta_r\}^T \{\epsilon_\tau\} \quad (17g)$$

$$\tau_\theta = \{\eta_r\}^T \{\epsilon_\tau\} \quad (17h)$$

where $\{\eta_\theta\}$ and $\{\eta_r\}$ are partial derivatives of $\{\eta\}$ with respect to θ and r , respectively. In view of Eqs. (15a) through (15c) and (17a) through (17h), the vector $\{\bar{\theta}\}$ defined in Eq. (2) can be expressed as

$$\{\bar{\theta}\} = [[B_1] \quad [B_2]] \{E\} \quad (18)$$

where

$$[B_1] = \frac{1}{2} \begin{bmatrix} 0 & 0 & z/r\{\eta_\theta\}^T \\ 0 & 0 & -z\{\eta_r\}^T \\ -1/r\{\eta_\theta\}^T & \langle S_1 \rangle & 0 \end{bmatrix} \quad (19a)$$

$$[B_2] = \frac{1}{2} \begin{bmatrix} -1/r\{\eta_\theta\}^T & 0 & -\{\eta\}^T \\ -\{\eta_r\}^T & \{\eta\}^T & 0 \\ 0 & -z/r\{\eta_\theta\}^T & \langle S_2 \rangle \end{bmatrix} \quad (19b)$$

$$\langle S_1 \rangle = 1/r\{\eta\}^T + \{\eta_r\}^T \quad (19c)$$

$$\langle S_2 \rangle = z/r\{\eta\}^T + z\{\eta_r\}^T \quad (19d)$$

Substituting for $\{\bar{\theta}\}$ from Eq. (18) into Eq. (1b), gives

$$\begin{aligned} \{\theta\} &= [T] [[B_1] \quad [B_2]] \{E\} \\ &= [[R_{21}] \quad [R_{22}]] \{E\} \end{aligned} \quad (20)$$

where, in view of $[T]$ defined in Eq. (2),

$$[R_{21}] = \frac{1}{2} \begin{bmatrix} 0 & 0 & \langle S_3 \rangle \\ 0 & 0 & \langle S_4 \rangle \\ -1/r\{\eta\}^T & \langle S_5 \rangle & 0 \end{bmatrix} \quad (21a)$$

$$[R_{22}] = \frac{1}{2} \begin{bmatrix} \langle S_6 \rangle & -s\{\eta\}^T & -c\{\eta\}^T \\ \langle S_7 \rangle & c\{\eta\}^T & -s\{\eta\}^T \\ 0 & -z/r\{\eta_\theta\}^T & \langle S_8 \rangle \end{bmatrix} \quad (21b)$$

$$\langle S_3 \rangle = z(c/r\{\eta_\theta\}^T + s\{\eta_r\}^T) \quad (21c)$$

$$\langle S_4 \rangle = z(s/r\{\eta_\theta\}^T - c\{\eta_r\}^T) \quad (21d)$$

$$\langle S_5 \rangle = 1/r\{\eta\}^T + \{\eta_r\}^T \quad (21e)$$

$$\langle S_6 \rangle = c/r\{\eta_\theta\}^T + s\{\eta_r\}^T \quad (21f)$$

$$\langle S_7 \rangle = s/r\{\eta_\theta\}^T + c\{\eta_r\}^T \quad (21g)$$

$$\langle S_8 \rangle = z/r\{\eta\}^T + z\{\eta_r\}^T \quad (21h)$$

Using Eqs. (18) and (20), the displacement components parallel to the global axes at any node on the actual system are related to the fictitious degrees-of-freedom of the super element by the following transformation:

$$\begin{Bmatrix} \{\delta\} \\ \{\theta\} \end{Bmatrix} = [R] \{E\} \quad (22)$$

where

$$[R] = \begin{bmatrix} [R_{11}] & [R_{12}] \\ [R_{21}] & [R_{22}] \end{bmatrix} \quad (23)$$

is the required transformation matrix relating the degrees-of-freedom of the super element to the degrees-of-freedom parallel to the global axes at a node of the actual system.

In view of Eqs. (1a), (1b), (14a), (14b), (21a) and (21b), Eq. (22) can be represented in terms of the sub-matrices as follows:

$$\begin{Bmatrix} \delta_x \\ \delta_y \\ \delta_z \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} = \begin{bmatrix} \langle A_1 \rangle & \langle A_2 \rangle & 0 & 0 & \langle A_5 \rangle & \langle A_6 \rangle \\ \langle B_1 \rangle & \langle B_2 \rangle & 0 & 0 & \langle B_5 \rangle & \langle B_6 \rangle \\ 0 & 0 & \langle C_3 \rangle & \langle C_4 \rangle & 0 & 0 \\ 0 & 0 & \langle D_3 \rangle & \langle D_4 \rangle & \langle D_5 \rangle & \langle D_6 \rangle \\ 0 & 0 & \langle E_3 \rangle & \langle E_4 \rangle & \langle E_5 \rangle & \langle E_6 \rangle \\ \langle F_1 \rangle & \langle F_2 \rangle & 0 & 0 & \langle F_5 \rangle & \langle F_6 \rangle \end{bmatrix} \begin{Bmatrix} \epsilon_u \\ \epsilon_v \\ \epsilon_\tau \\ \epsilon_w \\ \epsilon_\alpha \\ \epsilon_\beta \end{Bmatrix} \quad (24)$$

where the row matrices appearing in the pre-multiplier matrix on the right hand side of the equation can be obtained in terms of the partitioned sub-matrices of the transformation matrix $[R]$ given by Eqs. (19a), (19b), (21a) and (21b).

In Eq. (22), $\{\delta\}$ and $\{\theta\}$ include the actual translations and rotations of a node parallel to the global coordinates of a node of the actual system. These displacements can be divided into the in-plane and the out-of-plane deformations, as follows:

$$\{DIP\}^T = \langle \delta_x, \delta_y, \theta_z \rangle \quad (25a)$$

$$\{DOP\}^T = \langle \delta_z, \theta_x, \theta_y \rangle \quad (25b)$$

Thus, in view of Eqns. (25a) and (24), the in-plane displacement components can be represented by

$$\{DIP\} = [RIP] \{EIP\} \quad (26)$$

where

$$[RIP] = \begin{bmatrix} \langle A_1 \rangle & \langle A_2 \rangle & \langle A_3 \rangle & \langle A_6 \rangle \\ \langle B_1 \rangle & \langle B_2 \rangle & \langle B_3 \rangle & \langle B_6 \rangle \\ \langle F_1 \rangle & \langle F_2 \rangle & \langle F_3 \rangle & \langle F_6 \rangle \end{bmatrix} \quad (27a)$$

$$\{EIP\}^T = \langle \{\epsilon_u\} \ \{\epsilon_v\} \ \{\epsilon_w\} \ \{\epsilon_\theta\} \rangle \quad (27b)$$

Similarly, in view of Eq. (25b) and Eq. (24), the out-of-plane displacement components can be represented by

$$\{DOP\} = [ROP] \{EOP\} \quad (28)$$

where

$$[ROP] = \begin{bmatrix} \langle C_3 \rangle & \langle C_4 \rangle & 0 & 0 \\ \langle D_3 \rangle & \langle D_4 \rangle & \langle D_5 \rangle & \langle D_6 \rangle \\ \langle E_3 \rangle & \langle E_4 \rangle & \langle E_5 \rangle & \langle E_6 \rangle \end{bmatrix} \quad (29a)$$

$$\{EOP\}^T = \langle \{\epsilon_z\} \ \{\epsilon_x\} \ \{\epsilon_y\} \ \{\epsilon_\theta\} \rangle \quad (29b)$$

Finally, combining Eqs. (26) and (28), the displacement components parallel to the global coor-

dinate axes of a node of the actual structure are related to the fictitious degrees-of-freedom of the super element by the following equation:

$$\begin{Bmatrix} \{DIP\} \\ \{DOP\} \end{Bmatrix} = \begin{bmatrix} [RIP] & 0 \\ 0 & [ROP] \end{bmatrix} \begin{Bmatrix} \{EIP\} \\ \{EOP\} \end{Bmatrix} \quad (30)$$

This transformation represents an uncoupled set of two equations (the in-plane case and the out-of-plane case, respectively). The pre-multiplying matrix on the right hand side of Eq. (30) is the required transformation matrix which relates the degrees-of-freedom of an actual node to the fictitious degrees-of-freedom of the super element.

(2) Beam element transformation

Applying the transformations derived in Eq. (30) for the near node, "i", and the far node, "j", of a typical beam element, "k", of the structure, as shown in Fig. 2, gives

(i) For Near Node i:

$$\begin{Bmatrix} \{DIP\} \\ \{DOP\} \end{Bmatrix}_i^k = \begin{bmatrix} [RIP] & 0 \\ 0 & [ROP] \end{bmatrix}_i \begin{Bmatrix} \{EIP\} \\ \{EOP\} \end{Bmatrix}_i \quad (31)$$

(ii) For Far Node j:

$$\begin{Bmatrix} \{DIP\} \\ \{DOP\} \end{Bmatrix}_j^k = \begin{bmatrix} [RIP] & 0 \\ 0 & [ROP] \end{bmatrix}_j \begin{Bmatrix} \{EIP\} \\ \{EOP\} \end{Bmatrix}_j \quad (32)$$

The twelve degrees-of-freedom of the beam element "k" can be grouped in a single vector $\{\Delta\}^k$, as follows:

$$\{\Delta\}^k = \begin{Bmatrix} \begin{Bmatrix} \{DIP\} \\ \{DOP\} \end{Bmatrix}_i \\ \begin{Bmatrix} \{DIP\} \\ \{DOP\} \end{Bmatrix}_j \end{Bmatrix}^k \quad (33)$$

The displacement vectors on the right hand side of

the above equation can be rearranged to give

$$\{\Delta\}^k = \begin{Bmatrix} \{DI\} \\ \{DO\} \end{Bmatrix}^k \quad (34)$$

where

$$\begin{aligned} \{DI\}^k &= \begin{Bmatrix} \{DIP\}_i \\ \{DIP\}_j \end{Bmatrix}^k \quad \text{in-plane components} \\ &= [RI]^k \{EIP\} \end{aligned} \quad (35a)$$

$$\begin{aligned} \{DO\}^k &= \begin{Bmatrix} \{DOP\}_i \\ \{DOP\}_j \end{Bmatrix}^k \quad \text{out-of-plane components} \\ &= [RO]^k \{EOP\} \end{aligned} \quad (35b)$$

Using Eqs. (31) and (32), the pre-multipliers in Eqs (35a) and (35b) can be expressed as

$$[RI]^k = \begin{bmatrix} [RIP]_i \\ [RIP]_j \end{bmatrix}^k \quad (36a)$$

$$[RO]^k = \begin{bmatrix} [ROP]_i \\ [ROP]_j \end{bmatrix}^k \quad (36b)$$

Finally, substituting Eqs. (35a), (35b), (36a) and (36b) into Eq. (34), and in view of Eqs. (31), (32) and (33), gives the desired transformation relating the global degrees-of-freedom of a beam element to the fictitious degrees-of-freedom of the super element, as follows:

$$\{\Delta\}^k = \begin{bmatrix} [RI] & 0 \\ 0 & [RO] \end{bmatrix}^k \begin{Bmatrix} \{EIP\} \\ \{EOP\} \end{Bmatrix} \quad (37)$$

Formulation of the Stiffness Matrix of the Super Element

The total strain energy, U , of the structural system is obtained by summing the individual strain energy contributions of each beam element, i.e.,

$$U = \sum_{i=1}^{ne} U^k \quad (38)$$

where ne represents the total number of elements and U^k is the strain energy of typical k^{th} element, which is given by

$$U^k = \frac{1}{2} \{\Delta\}^{kT} [K]^k \{\Delta\}^k \quad (39)$$

where $[K]^k$ is the standard (12 x 12) global stiffness matrix of the k^{th} beam element. Substituting Eq. (37) into Eq. (39), the strain energy of the k^{th} element is given by

$$\begin{aligned} U^k &= \frac{1}{2} \{EIP\}^T [EKIP]^k \{EIP\} + \\ &\quad \frac{1}{2} \{EOP\}^T [EKOP]^k \{EOP\} \end{aligned} \quad (40)$$

In the above equation, the strain energy contribution by the k^{th} element is considered to be the algebraic sum of the in-plane and the out-of-plane components. Thus, the stiffness matrices $[EKIP]^k$ and $[EKOP]^k$, respectively, represent the in-plane and out-of-plane stiffness contributions by a beam element to the stiffness matrix of the super element. These are given by

$$[EKIP]^k = [RI]^{kT} [KI]^k [RI]^k \quad (41a)$$

$$[EKOP]^k = [RO]^{kT} [KO]^k [RO]^k \quad (41b)$$

where $[KI]^k$ and $[KO]^k$ represent the standard (6 x 6) stiffness matrices of the k^{th} beam element corresponding to the in-plane and the out-of-plane deformations, respectively.

Thus, substituting Eq. (40) into (38), the total strain energy of the whole system is given by

$$\begin{aligned} U &= \frac{1}{2} \{EIP\}^T [KIP] \{EIP\} + \\ &\quad \frac{1}{2} \{EOP\}^T [KOP] \{EOP\} \end{aligned} \quad (42)$$

where the in-plane and the out-of-plane pseudo stiffness matrices, $[KIP]$ and $[KOP]$, respectively, of the super element are given by

$$[KIP] = \sum_{i=1}^{ne} [EKIP]^k \quad (43a)$$

$$[KOP] = \sum_{i=1}^{ne} [EKOP]^k \quad (43b)$$

Formulation of the Mass Matrix of the Super Element

The total kinetic energy, T , of the structural system is obtained by summing the individual kinetic energy contributions of each beam element, i.e.,

$$T = \sum_{i=1}^{ne} T^k \quad (44)$$

where T^k is the kinetic energy of a typical k^{th} element, which is given by

$$T^k = \frac{1}{2} \{\dot{\Delta}\}^{kT} [M]^k \{\dot{\Delta}\}^k \quad (45)$$

where a dot over a variable denotes its derivative with respect to time, and $[M]^k$ is the standard (12 x 12) global consistent mass matrix of the k^{th} beam element. Taking the first derivative of $\{\Delta\}^k$ in Eq. (37) with respect to time, gives

$$\{\dot{\Delta}\}^k = \begin{bmatrix} [RI] & 0 \\ 0 & [RO] \end{bmatrix}^k \begin{Bmatrix} \{\dot{EIP}\} \\ \{\dot{EOP}\} \end{Bmatrix} \quad (46)$$

Substituting this equation into Eq. (45), the kinetic energy of the k^{th} element is given by

$$T^k = \frac{1}{2} \{\dot{EIP}\}^T [EMIP]^k \{\dot{EIP}\} + \frac{1}{2} \{\dot{EOP}\}^T [EMOP]^k \{\dot{EOP}\} \quad (47)$$

In the above equation, the kinetic energy contribution by the k^{th} element is treated to be the algebraic

sum of the in-plane and the out-of-plane components. Thus the mass matrices $[EMIP]^k$ and $[EMOP]^k$, respectively, represent the in-plane and out-of-plane mass contributions by a beam element to the mass matrix of the super element. These are given by

$$[EMIP]^k = [RI]^{kT} [MI]^k [RI]^k \quad (48a)$$

$$[EMOP]^k = [RO]^{kT} [MO]^k [RO]^k \quad (48b)$$

where $[MI]^k$ and $[MO]^k$ represent the standard (6 x 6) mass matrices of the k^{th} beam element corresponding to the in-plane and the out-of-plane deformations, respectively.

Thus, substituting Eq. (47) into Eq. (44), the total kinetic energy of the whole system is given by

$$T = \frac{1}{2} \{\dot{EIP}\}^T [MIP] \{\dot{EIP}\} + \frac{1}{2} \{\dot{EOP}\}^T [MOP] \{\dot{EOP}\} \quad (49)$$

where the in-plane and the out-of-plane pseudo mass matrices, $[MIP]$ and $[MOP]$, respectively, of the super element are given by

$$[MIP] = \sum_{i=1}^{ne} [EMIP]^k \quad (50a)$$

$$[MOP] = \sum_{i=1}^{ne} [EMOP]^k \quad (50b)$$

Transformation of the Load Vector

Since the system dynamic load vector is specified with reference to the degrees-of-freedom of the actual system, this has to be transformed to conform to the degrees-of-freedom of the super element. In view of the in-plane and the out-of-plane coordinates defined earlier, the equation of motion of a beam element "k" of the actual undamped system is given by

$$\begin{bmatrix} [MI] & 0 \\ 0 & [MO] \end{bmatrix}^k \begin{Bmatrix} \{\ddot{DI}\} \\ \{\ddot{DO}\} \end{Bmatrix}^k + \begin{bmatrix} [KI] & 0 \\ 0 & [KO] \end{bmatrix}^k \begin{Bmatrix} \{DI\} \\ \{DO\} \end{Bmatrix}^k = \begin{Bmatrix} \{PIN\} \\ \{POUT\} \end{Bmatrix}^k \quad (51)$$

where $\{PIN\}^k$ and $\{POUT\}^k$ are the load vectors corresponding to the in-plane and the out-of-plane deformations, $\{DI\}^k$ and $\{DO\}^k$, respectively, of the k^{th} beam element. It can be seen from this equation that the in-plane and the out-of-plane equations of motion are uncoupled and can be treated separately as follows:

(1) For the in-plane case

From Eq. (51) the in-plane equation of motion can be written as

$$[M]^k \{\ddot{DI}\}^k + [K]^k \{DI\}^k = \{PIN\}^k \quad (52)$$

Substituting Eq. (35a) into Eq. (52) and pre-multiplying the resulting equation throughout by $[R]^T$, reduces the in-plane element equations of motion to

$$[EMIP]^k \{\ddot{EIP}\} + [EKIP]^k \{EIP\} = \{EPIP\}^k \quad (53)$$

where $[EMIP]^k$ and $[EKIP]^k$ have been defined in Eqs. (48a) and (41a), respectively, and the transformed load vector, $\{EPIP\}^k$ for the k^{th} element, is given by

$$\{EPIP\}^k = [R]^k \{PIN\}^k \quad (54)$$

Thus, the total in-plane load vector for the super element is given by

$$\{PIP\} = \sum_{i=1}^{nc} \{EPIP\}^k \quad (55)$$

(2) For the out-of-plane case

From Eq. (52) the out-of-plane equation of motion can be written as

$$[MO]^k \{\ddot{DO}\}^k + [KO]^k \{DO\}^k = \{POUT\}^k \quad (56)$$

Substituting Eq. (35b) into Eq. (56) and pre-multiplying throughout by $[RO]^T$, reduces the out-of-plane element equations of motion to

$$[EMOP]^k \{\ddot{EOP}\}^k + [EKOP]^k \{EOP\} = \{EPOP\}^k \quad (57)$$

where $[EMOP]^k$ and $[EKOP]^k$ have been defined in Eq. (49b) and (41b), respectively, and the transformed load vector, $\{EPOP\}^k$ for the k^{th} element, is given by

$$\{EPOP\}^k = [RO]^k \{POUT\}^k \quad (58)$$

Thus, the total out-of-plane load vector for the super element is given by

$$\{POP\} = \sum_{i=1}^{nc} \{EPOP\}^k \quad (59)$$

Transient Response Analysis

In the method presented, the response analysis of the super element is performed first. These results are used to obtain the response characteristics of the actual system. The undamped equations of motion for the super element for the in-plane and the out-of-plane can be represented as

(1) The in-plane case

$$[MIP] \{\ddot{EIP}\} + [KIP] \{EIP\} = \{PIP\} \quad (60)$$

(2) The out-of-plane case

$$[MOP] \{\ddot{EOP}\} + [KOP] \{EOP\} = \{POP\} \quad (61)$$

The above two equations can be solved by any standard procedure to give the response of the super element. Then the response of the i^{th} node of the actual system is obtained from

$$\{DIP\}_i = [RIP]_i \{EIP\} \quad (62)$$

for the in-plane case, and

$$\{DOP\}_i = [ROP]_i \{EOP\} \quad (63)$$

for the out-of-plane case. The matrices $[RIP]_i$ and $[ROP]_i$ have been defined earlier by Eqs. (27a) and (29a), respectively.

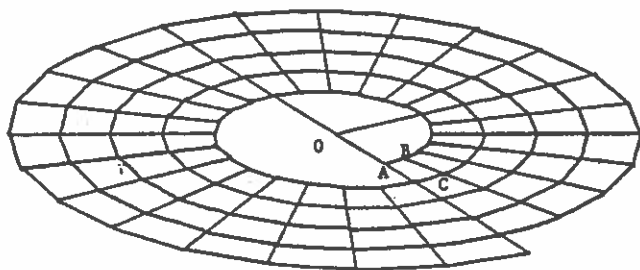
3. Numerical Results

The configuration of the spiral structure analyzed is shown in Fig. 4. The nodes of the structure lie on

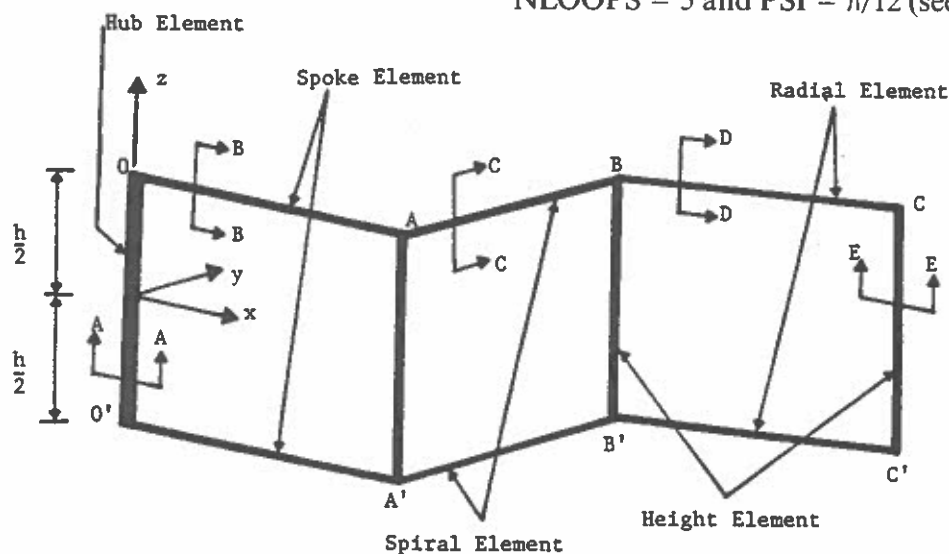
an Archimedian spiral, the equation of which is taken as

$$R = 5 (1 + 0.0955 \theta) \quad (64)$$

where R = radius in ft at a node point and θ = angle in radians a node makes at the origin in cylindrical coordinates. Equation (64) corresponds to a five loop Archimedian spiral with a

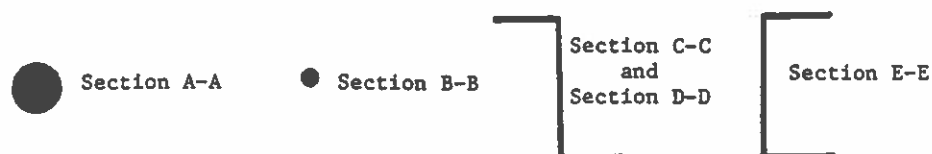


(a) Top view



$O'A'B'C'$ are below $OABC$ and h = height of the spiral

(b) Configuration of element types



(c) Cross-sectional shapes of typical elements

Fig. 4. Geometric configuration and sectional shapes used for the spiral structure analyzed

minimum radius = 5 ft and a maximum radius = 20 ft. The height dimensions of the spiral is taken to be 4 ft. As shown in Fig. 4, the structural system comprises of the the following five types of elements: hub, spoke, spiral, radial and height elements. The geometry and material data used for these elements are given in Table 1. The structure is fixed at the hub and the spoke elements do not have any displacements at their points of attachment to the spiral. All other nodes are free. The discretized conventional finite element model of the structural system, so obtained, will have 244 nodes (=NNODES), 560 elements and 1,440 degrees-of-freedom.

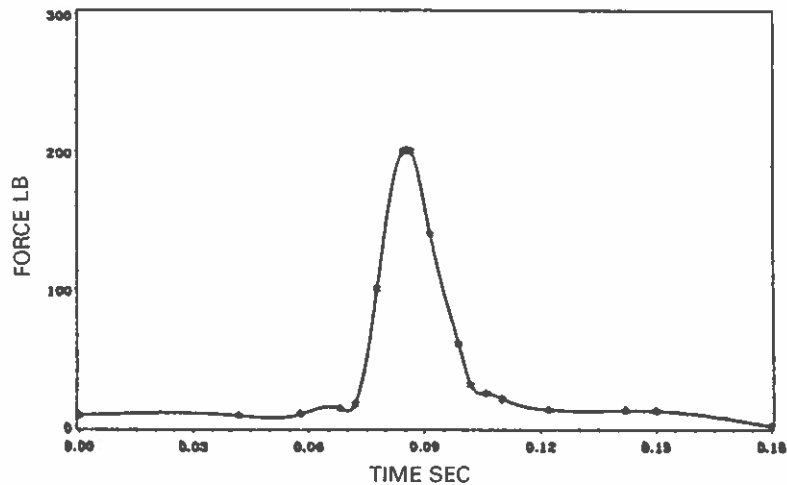
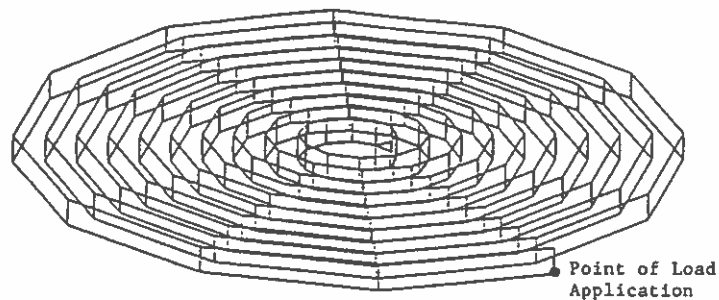
The dynamic load is taken to act on the structure at the last numbered node, as shown in Fig. 5(a). This dynamic load has components acting along the global x -, y - and z -axis, and the time dependent history taken for these components is shown in Figs. 5(b) and 5(c).

The response analysis of the structure with NLOOPS = 5 and PSI = $\pi/12$ (see the footnotes b

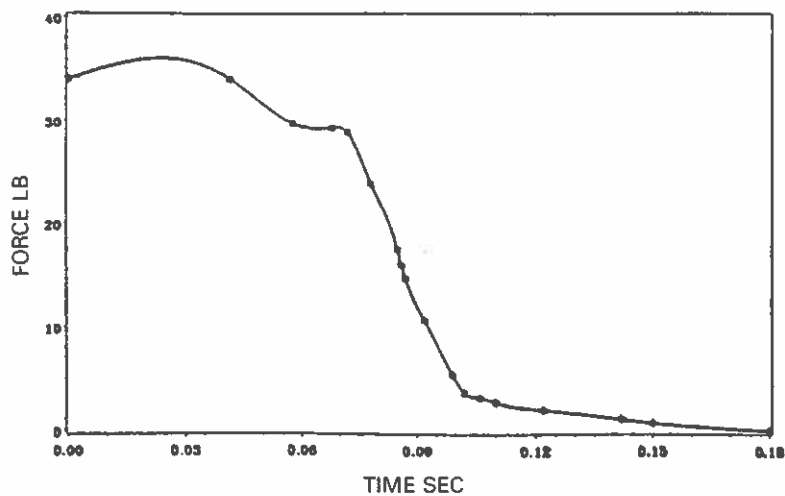
and c in Table 1 for definition of PSI and NLOOPS, respectively) was carried out using the method presented and the fourth-order Runge-Kutta method with Gill's modification³ for solving the equations of motion in the time domain. This analysis was performed using an Alliant FX/4

parallel processor computer. To verify the results obtained, a conventional finite element model of the structure was analyzed by using the computer software package MSC/NASTRAN.⁴ This analysis was performed on an IBM 3081 computer. The total time for which the response was computed

(a) Loaded node



(b) Load versus time variation for the x- and y-direction load components



(c) Load versus time variation for the z-direction load component

Fig. 5. Load data used for the spiral structure analyzed

Table 1. Data Used for the Geometric Dimensions and Material Properties for the Spiral Structure

Item	Element Location				
	Hub	Spoke	Along the Spiral	Along Radial Direction	Along Height Dimension
Length (ft)	4	R_i^a	L_k^b	L_r^d	4
Cross-Sectional Area (ft ²)	0.5	3.47×10^{-3}	3.47×10^{-3}	3.47×10^{-3}	3.47×10^{-3}
Cross-Sectional Moment of Inertia					
I_y (ft ⁴)	0.1	3.66×10^{-3}	3.66×10^{-3}	3.66×10^{-3}	1.73×10^{-3}
I_z (ft ⁴)	0.1	5.30×10^{-3}	5.30×10^{-3}	5.30×10^{-3}	5.30×10^{-3}
Modulus of Elasticity (lb/ft ²)	4.32×10^6	4.32×10^6	1.44×10^6	1.44×10^6	1.44×10^6
Modulus of Rigidity (lb/ft ²)	1.73×10^6	1.73×10^6	5.41×10^5	5.41×10^5	5.41×10^5
Poisson's Ratio	0.3	0.3	0.33	0.33	0.33
Mass Density (lb/ft ³)	501.12	501.12	172.8	172.8	172.8
Number of Elements (NLE)	1	6	2 (NNPL) (NLOOPS) ^c	2 (NNPL) (NLOOPS-1)	(NNODES/2)-1 ^e

^a R_i calculated from Eq. (64) with $\theta = 0, \pi/2$ and π .

^b $L_k^2 = R_i^2 + R_{i+1}^2 - 2 R_i R_{i+1} \cos(\text{PSI})$, where subscripts are i and $i+1$ denote the near and far nodes of the element, respectively, PSI is the angle subtended at the origin by two successive nodes along the spiral, and R_i and R_{i+1} are calculated using Eq. (64).

^cNNPL = number of nodes per loop = $2\pi/\text{PSI}$, and NLOOPS = total number of loops considered.

^d $L_r = R_{i+\text{NNPL}} - R_i$, where subscripts i and $i + \text{NNPL}$ denote the near and far nodes of the element, respectively, and R_i and $R_{i+\text{NNPL}}$ are calculated using Eq. (64).

^eNNODES = total number of nodes.

was taken as 0.18 sec. It was seen that dividing the time domain in 20, 40, 80 and 160 equal parts did not produce any significant change in the results. The acceleration response results obtained in the z loaded direction by taking $n = 1, 4, 8, 10$ and 12 in the method presented is shown in Fig. 6. Similar response plots for displacement, velocity and acceleration in the loaded directions, are presented in Ref. 5, and are not presented here for space considerations. Table 2 gives the degrees-of-freedom of the super element for the values of " n " chosen and the respective CPU (Central Processor Unit) time required for each response analysis. From the plots shown in Fig. 6 it can be seen that the results

converges as n is increased. The acceleration response obtained by taking $n = 12$ in the method presented is compared in Fig. 7 with the acceleration response obtained by analyzing the 1,440 degrees-of-freedom conventional finite element model of the structure by the MSC/NASTRAN computer software package. Similar response plots for displacement, velocity and acceleration for all loaded nodes are presented in Ref. 5. The difference between the results obtained for acceleration response by the two methods was the largest in comparison to the velocity and the displacement responses.

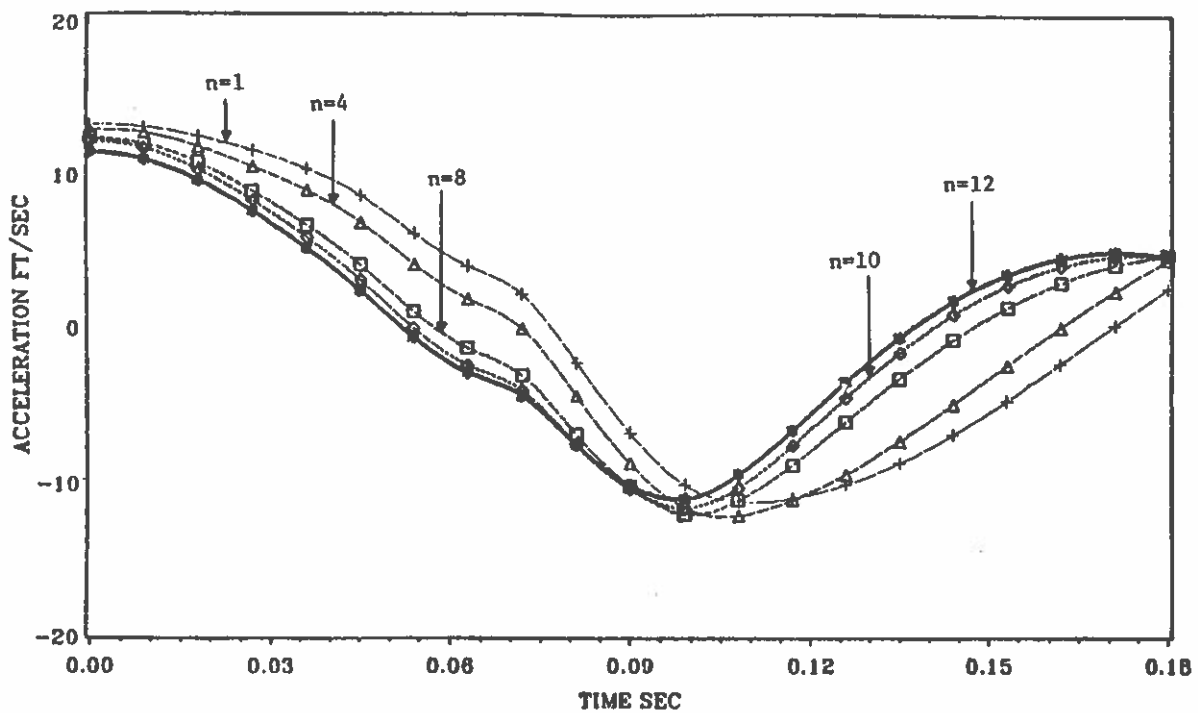


Fig. 6. Acceleration response in the loaded z-direction obtained by the method presented for different values of n

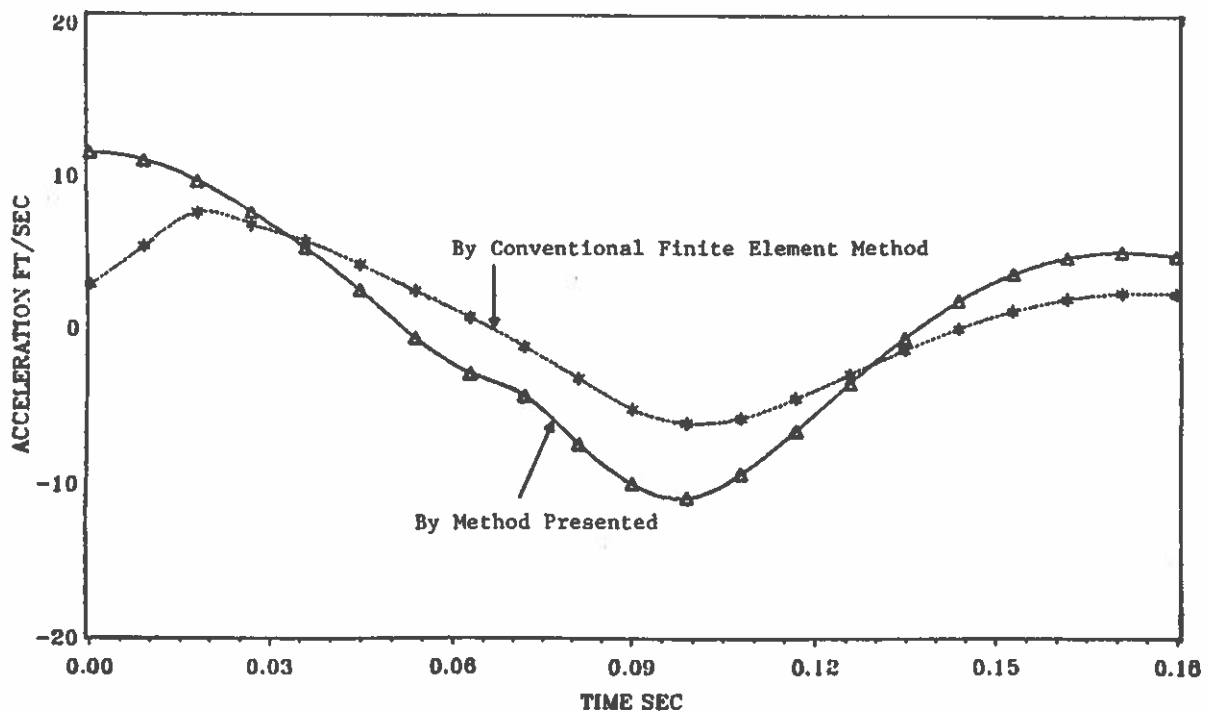


Fig. 7. Comparison of acceleration response in the loaded z-direction by the method presented and the conventional finite element method

Table 2. Degrees-of-Freedom of the Super Element with Increasing Number of Terms Taken in the Assumed Trigonometric Series and the CPU Time Taken for the Analysis.

n	Super Element Degrees-of-Freedom	CPU Time (sec)
1	16	14.7
4	64	84.3
8	128	172.0
10	160	448.3
12	192	639.8

4. Conclusions

The analysis of linear elastic latticed structures which have the height dimension very small as compared to its overall length and width dimensions can be analyzed by embedding the structure in its continuum and choosing appropriate kinematically admissible functions to describe the deformation of the middle plane of the continuum. This deformation can be expressed in terms of a trigonometric series, the unknown coefficients of which will represent the degrees-of-freedom of the pseudo structure. In this paper, this technique of analysis has been used to analyze a structure made up of an assemblage of beam elements. For the form of kinematically admissible function chosen to model the deformation of the middle plane of the continuum, it was found that the response results obtained were in close agreement with those obtained by using the conventional finite element method. It was also found that a larger number of terms are needed in the assumed kinematically admissible function to obtain a higher degree of accuracy for velocity and acceleration responses in comparison to that needed for the displacement response. In all cases, for the specified loading, it was found that the response solutions tended to converge towards the conventional finite element solutions in a upper

bound fashion.

The method presented in this paper will be economical for a problem in which the discretized finite element model involves a very large number of degrees-of-freedom. In such a situation, it may be possible to solve this problem by the method presented by using a much smaller number of terms in the kinematically admissible function chosen (say 20). If n = the number of terms in the kinematically admissible function, then the size of the resulting matrices in the method developed will be of the order $8n \times 8n$ for the in-plane and the out-of-plane equations of motion. In such a situation, not only can the problem be accommodated in the computer core memory, but significant savings in computational time can be obtained. The savings in computational time would be even larger for problems where modal superposition procedures are used for response analysis.

The method presented in this paper is amenable for parallel processing, which can further reduce computational time. For the problem presented, it was found that by using parallel processing in the computer program, the CPU time was reduced by about 3.5 time using an Alliant FX/4 computer. These savings in computational time can be quite considerable for analysis of very large latticed structures.

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